MATH2060 TUT04

16. Let $I \subseteq \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be differentiable on I, and suppose $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

Give an example where this limit exists, but the function does not have a second derivative at a.

Ans: f diff. on $I \Rightarrow f$ cts on I

So $F(h) := f(a+h) - 2f(a) + f(a-h)$ is diff. in a nbhd of 0

and $\lim_{h\to 0} F(h) = 0$

OTOH, $G(h) := h^2$ is clearly diff. $\forall\, h \in \mathbb{R}$

and $G(h) \neq 0 \;\; \forall\, h \neq 0$, $\lim_{h\to 0} G(h) = 0$

By L'Hopital's Rule

$$\lim_{h\to 0} \frac{F(h)}{G(h)} = \lim_{h\to 0} \frac{F'(h)}{G'(h)} \quad \text{(provided RHS exists)}$$

$$= \lim_{h\to 0} \frac{f'(a+h) - f'(a-h)}{2h}$$

$$= \frac{1}{2} \lim_{h\to 0} \left[\frac{f'(a+h) - f'(a)}{h} + \frac{f'(a-h) - f'(a)}{-h} \right]$$

$$= \frac{1}{2} \left(f''(a) + f''(a) \right) = f''(a)$$

Take $a = 0$, $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$. $\Rightarrow f'(x) = \begin{cases} 2x, & x > 0 \\ 0, & x = 0 \\ -2x, & x < 0 \end{cases}$

Then f is diff. on $\mathbb{R}$ and $\lim_{h\to 0} \frac{f(0+h) - 2f(0) + f(0-h)}{h^2} = \lim_{h\to 0} 0 = 0$

However $f''(0)$ DNE since $\lim_{x\to 0^+} \frac{f'(x) - f'(0)}{x} = 2$ while $\lim_{x\to 0^-} \frac{f'(x) - f'(0)}{x} = -2$

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Thm 6.4.7 (Newton's Method)

Let
- $f : [a,b] \to \mathbb{R}$ twice differentiable $(a<b)$
- $f(a)f(b) < 0$
- $\exists$ constants $m > 0$, $M \geq 0$ s.t.

$$|f'(x)| \geq m > 0 \text{ and } |f''(x)| \leq M, \quad \forall x \in [a,b].$$

Need to find $\delta > 0$ so small s.t. $I^* := [r-\delta, r+\delta] \subseteq [a,b]$ and $\delta < 1/K$.

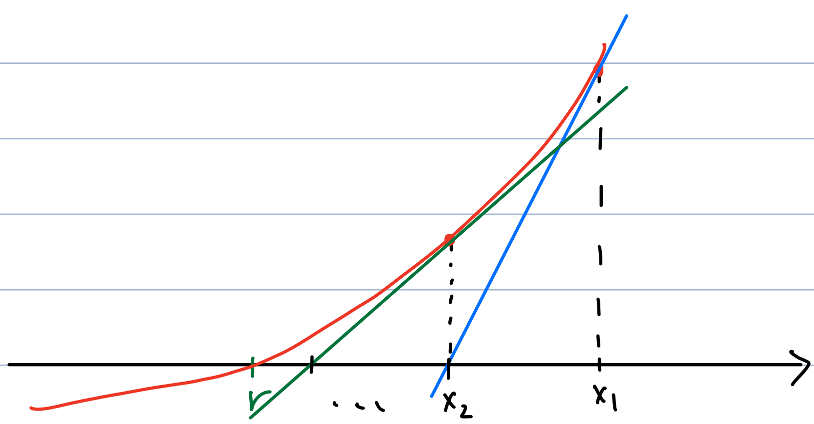
Then $\exists$ a subinterval $I^* \subseteq [a,b]$
- containing a zero r of s.t.
- $\forall x_1 \in I^*$, the sequence (x_n) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \forall n = 1,2,3,\dots$$

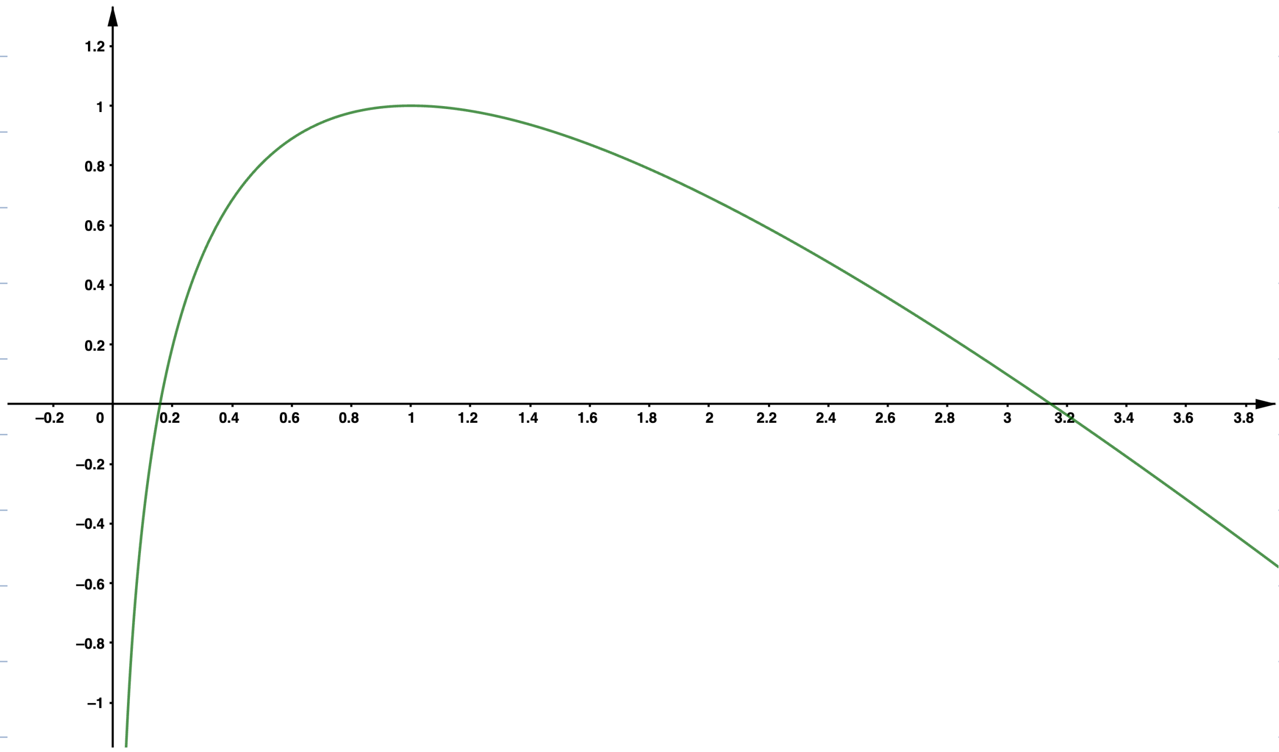
belongs to I^*, and
- $\lim_{n\to\infty} x_n = r$

Moreover
- $|x_{n+1} - r| \leq K|x_n - r|^2 \quad \forall n = 1,2,3,\dots$

where $K = \frac{M}{2m}$.



22. The equation $\ln x = x - 2$ has two solutions. Approximate them using Newton's Method. What happens if $x_1 := \frac{1}{2}$ is the initial point?



Ans: Let $f(x) = \ln x - x + 2$.

Then f is twice diff. on $(0, \infty)$ with

$$f'(x) = \frac{1}{x} - 1, \qquad f''(x) = -\frac{1}{x^2}$$

If $x_1 = \frac{1}{2}$, then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= \frac{1}{2} - \frac{\ln\left(\frac{1}{2}\right) + \frac{3}{2}}{1}$$

$$= \ln 2 - 1 < 0$$

Newton's Method cannot proceed since $f(x_2)$ is not defined. //

22. The equation $\ln x = x - 2$ has two solutions. Approximate them using Newton's Method. What happens if $x_1 := \frac{1}{2}$ is the initial point?

Ans:

Since $f(0.1) \approx -0.4026$ and $f(0.2) \approx 0.1906 > 0$

the Intermediate Value Thm implies that $\exists\, r \in I := [0.1, 0.2]$ s.t. $f(r) = 0$.

Let $x_1 = 0.2$, $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \quad \forall n \geq 1$.

Then $x_2 \approx 0.152359$, $x_3 \approx 0.158594$, $x_4 \approx 0.158594$,

$x_5 \approx 0.158594$, $x_6 \approx 0.158594$, ...

Does $x_n \to r$? How accurate is it ?

Note $\left.\begin{array}{l} m := \min_{x \in I} |f'(x)| = \frac{1}{0.2} - 1 = 4, \\ M := \max_{x \in I} |f''(x)| = \frac{1}{0.1^2} = 100 \end{array}\right\} \Rightarrow \begin{array}{l} K := \frac{M}{2m} = \frac{25}{2} \\ 1/K = 0.08 \end{array}$

Let $I^* = [r - \delta, r + \delta] = [2r - 0.2, 0.2]$.

Then $I^* \subseteq I$ $\quad (\because r \in (0.15, 0.2)$ as $f(0.15) \approx -0.0471 < 0)$

and $0 < \delta \leq 0.05 < 1/K$.

By Newton's method (and its proof),

1) $x_n \in I^* \quad \forall n \qquad (\because x_1 \in I^*)$

2) $x_n \to r$

3) $|x_{n+1} - r| \leq K |x_n - r|^2 \quad \forall n \in \mathbb{N}$.

In particular, $e_n := K(x_n - r)$ satisfies

$$|e_{n+1}| \leq |e_n|^2 \leq \cdots \leq |e_1|^{2^n}$$

So $|e_6| \leq |e_1|^{2^5}$

$$\Rightarrow |x_6 - r| \leq 0.08 (0.05/0.08)^{32} \approx 0.24 \times 10^{-7} < 0.5 \times 10^{-6}$$

Thus $r \approx x_6 \approx 0.158594$, cor. to 6 d.p.

22. The equation $\ln x = x - 2$ has two solutions. Approximate them using Newton's Method. What happens if $x_1 := \frac{1}{2}$ is the initial point?

Ans: Since $f(3) = \ln 3 - 1 > 0 \quad (\approx 0.0986)$

$f(4) = \ln 4 - 2 < 0 \quad (\approx -0.6137)$,

the Intermediate Value Thm implies that $\exists\, s \in I' := [3, 4]$ s.t. $f(s) = 0$

Let $x_1 = 3$, $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \quad \forall n \geq 1$.

Then $x_2 \approx 3.147918$, $x_3 \approx 3.146193$, $x_4 \approx 3.146193$, ...

Again, we can show that $x_n \to s$ and estimate the error.

Note $m := \min_{x \in I'} |f'(x)| = \left|\frac{1}{4} - 1\right| = \frac{3}{4}$

$M := \max_{x \in I'} |f''(x)| = \frac{1}{3^2} = \frac{1}{9}$

$\Rightarrow K := \dfrac{M}{2m} = \dfrac{2}{3}$, $\quad 1/K = 1.5$

Let $I^{**} = [s - \delta, s + \delta] = [3, 2s - 3]$.

Then $I^{**} \subseteq I'$ $\quad (\because s \in (3, 3.5)$ as $f(3.5) \approx -0.24 < 0)$

and $0 < \delta \leq 0.5 < 1/K$.

By Newton's method (and its proof),

1) $x_n \in I^{**} \quad \forall n \quad (\because x_1 \in I^{**})$

2) $x_n \to s$

3) $|x_{n+1} - s| \leq K |x_n - s|^2 \quad \forall n \in \mathbb{N}$.

Def 1) A fcn $f:[a,b]\to\mathbb{R}$ is said to be Riemann integrable on $[a,b]$ if $\exists\, L\in\mathbb{R}$ s.t. $\forall\, \varepsilon>0$, $\exists\, \delta_\varepsilon>0$ s.t. $\forall$ tagged partition $\dot{P}$ of $[a,b]$ with $\|\dot{P}\|<\delta_\varepsilon$

$$|S(f;\dot{P})-L|<\varepsilon$$

For $\dot{P}=\{[x_{i-1},x_i],t_i\}_{i=1}^n$, $\|\dot{P}\|=\max\{|x_i-x_{i-1}| : i=1,\dots,n\}$

$$S(f;\dot{P})=\sum_{i=1}^n f(t_i)(x_i-x_{i-1})$$

2) $\mathcal{R}[a,b]:=$ set of all Riemann integrable fcns on $[a,b]$

3) The number L is uniquely determined and is denoted by

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)\,dx$$

3. Show that $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with norm $||\dot{\mathcal{P}}|| \le \delta_\varepsilon$, then $|S(f; \dot{\mathcal{P}}) - L| \le \varepsilon$. (*)

Pf: "$\Rightarrow$" Suppose $f \in \mathcal{R}[a,b]$. Write $L = \int_a^b f$. Let $\varepsilon > 0$.

By def., $\exists\, \delta_\varepsilon > 0$ s.t.

$\forall$ tagged partition $\dot{P}$ of $[a,b]$ with $\|\dot{P}\| < \delta_\varepsilon$,

$$|S(f;\dot{P}) - L| < \varepsilon$$

Take $\delta_\varepsilon' := \delta_\varepsilon/2$ (>0).

Then $\forall$ tagged partition $\dot{P}$ of $[a,b]$ with $\|\dot{P}\| \le \delta_\varepsilon'$,

we have $\|\dot{P}\| < \delta_\varepsilon$ and hence

$$|S(f;\dot{P}) - L| < \varepsilon$$

$$\Rightarrow \quad |S(f;\dot{P}) - L| \le \varepsilon .$$

So (*) holds.

"$\Leftarrow$" Suppose (*) holds.

Then $\exists\, L \in \mathbb{R}$ s.t. $\forall\, \varepsilon > 0$ ($\varepsilon/2 > 0$), $\exists\, \delta_\varepsilon > 0$ s.t.

$\forall$ tagged partition $\dot{P}$ of $[a,b]$ with $\|\dot{P}\| \le \delta_\varepsilon$ ($\delta_{\varepsilon/2}$).

$$|S(f;\dot{P}) - L| \le \varepsilon/2$$

So, $\forall\, \varepsilon > 0$, $\exists\, \delta_{\varepsilon/2} > 0$ s.t.

$\forall$ tagged partition $\dot{P}$ of $[a,b]$ with $\|\dot{P}\| < \delta_{\varepsilon/2}$,

$$|S(f;\dot{P}) - L| \le \varepsilon/2 < \varepsilon .$$

Thus $f \in \mathcal{R}[a,b]$. //

5. Let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n}$ be a tagged partition of $[a, b]$ and let $c_1 < c_2$.
   (a) If u belongs to a subinterval I_i whose tag satisfies $c_1 \le t_i \le c_2$, show that $c_1 - ||\dot{\mathcal{P}}|| \le u \le c_2 + ||\dot{\mathcal{P}}||$.
   (b) If $v \in [a, b]$ and satisfies $c_1 + ||\dot{\mathcal{P}}|| \le v \le c_2 - ||\dot{\mathcal{P}}||$, then the tag t_i of any subinterval I_i that contains v satisfies $t_i \in [c_1, c_2]$.

Ans: a) Write $I_i = [x_{i-1}, x_i]$

Then $x_{i-1} \le u, t_i \le x_i$

So $|u - t_i| \le x_i - x_{i-1} \le ||\dot{P}||$

$\Rightarrow \quad t_i - ||\dot{P}|| \le u \le t_i + ||\dot{P}||$

$\Rightarrow \quad c_1 - ||\dot{P}|| \le u \le c_2 + ||\dot{P}||$

b) Since $P = \{I_i\}_{i=1}^{n}$ is a partition of $[a,b]$,

$\exists\ i \in \{1, \dots, n\}$ s.t. $v \in I_i$

Replace the tag of I_i in $\dot{P}$ by v to get a new tagged partition $\dot{Q}$.

Then $||\dot{Q}|| = ||\dot{P}||$.

Now, $t_i \in I_i$ whose tag satisfies $c_1 + ||\dot{P}|| \le v \le c_2 - ||\dot{P}||$.

By a),

$$(c_1 + ||\dot{P}||) - ||\dot{Q}|| \le t_i \le (c_2 - ||\dot{P}||) + ||\dot{Q}||$$

$$c_1 \le t_i \le c_2$$

i.e. $t_i \in [c_1, c_2]$

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